Fixed Point Theorem for a Pair of Self Maps Satisfying a General Contractive Condition of Exponential Type

Priyanka Sharma[1], Dr. D. B. Ojha[2]

[1]Research Scholar, Mewar University, Chittorgarh, Rajasthan

[2]Department of Mathematics, Mewar University, Chittorgarh, Rajasthan

ABSTRACT

In this paper, the establishment of a fixed point theorem for a pair of self maps satisfying a general contractive condition of exponential type will be proposed. We will use complete metric space to prove the result of the following theorems. We will also show that Cauchy sequence is convergent in complete metric space. The results obtained in the complete metric space by using the concept of pair of self maps are very interesting. We prove common fixed point theorems for pair of self maps in complete metric space by using the contractive condition. We also find an affirmative answer in complete metric space to the problem of "Banach- cacciopoli".

Keywords: Complete metric space, Lebesgue integral, Exponential terms, Contractive condition.

1. INTRODUCTION

The first well known result of Banach-cacciopoli was on fixed points for contractive map, **[1][2][4].** In general setting of complete metric space, smart presented the following result as well as **[1]-[3]**.

Theorem 1.1: Let (X,d) be a complete metric space, $c \in [0,1)$ and let $T : X \to X$ be a map s.t. for each $x, y \in X$,

$$d\left(T_x, T_y\right) \le cd\left(x, y\right)$$

Then T has a unique fixed point $z \in X$ s.t. for each $x \in X, \lim_{n\to\infty} T^n x = z$.

After this classical result, many theorems dealing with maps satisfying various types of contractive inequalities have been established **[2], [5]-[10], [14]**, and obtained the following theorem as,

Theorem 1.2: Let (X, d) be a complete metric space, $c \in [0,1)$ and let $T : X \to X$ be a map such that for every $x, y \in X$,

$$e^{d\left(T_x, T_y\right)} \le ce^{(x,y)}$$

Where $\phi : R^+ \to R^+$ is a lebesgue- integrable map which is summable, positive and such that $e^{\varepsilon} > 0$ for each $\varepsilon > 0$.

Then T has a unique fixed point $z \in X$ and for each $x \in X, \lim_{n\to\infty} T^n x = z$.

In paper **[2]**, some fixed point theorems for a self map satisfying a general that one can generalize other results related to contractive conditions of some kind, such as in **[6]- [8]**.

The main object of this paper is to obtain some results for a pair of self maps satisfying a general contractive condition of exponential type.
Throughout this paper, N= Set of natural numbers.

2. MAIN RESULTS

Theorem 2.1: Let (X, d) be a complete metric space. Let a_i (i=1, 2,..,5) be positive real numbers satisfying $\sum_{i=1}^{5} a_i < 1$, T_1 and T_2 be a pair of self maps of the metric space X such that for every $x, y \in X$,

$$e^{d(T_1x, T_2y)} \le a_1 e^{d(x,y)} + a_2 e^{d(x,T_1x)} + a_3 e^{d(y,T_2y)} + a_4 e^{d(x,T_2y)} + a_5 e^{d(y,T_1x)} \qquad \text{……...(2.1)}$$

Where $\phi : R^{+} \rightarrow R^{+}$ is a lebesgue- integrable map which is summable, positive and such that $e^{\varepsilon} > 0$ for every $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in X$.

Proof: Let x_0 be any point of X.

Define $x_{2n-1} = T_1 x_{2n-2}$

$x_{2n} = T_2 x_{2n-1}$ where $n \in N$.

We claim that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0 \qquad \text{.........(2.2)}$$

To prove (2.2), we require showing that

$$e^{d(x_n, x_{n+1})} \le r^n e^{d(x_0, x_1)} \quad \text{Where } r = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}$$

For this, by interchanging x with y and T_1 with T_2 in (2.1), we obtain

$$e^{d(T_2 y, T_1 x)} \le a_1 e^{d(y,x)} + a_2 e^{d(y, T_2 y)} + a_3 e^{d(x, T_1 x)} + a_4 e^{d(y, T_1 x)} + a_5 e^{d(x, T_2 y)} \qquad \text{........(2.3)}$$

Now from (2.1), (2.3) and using symmetric property, we obtain

$$e^{d(T_1 x, T_2 y)} \le a_1 e^{d(x,y)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x, T_1 x)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(y, T_2 x)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x, T_2 y)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(y, T_1 x)} \qquad \text{........(2.4)}$$

Using (2.4) for odd n, we obtain

$$e^{d(x_n, x_{n+1})} = e^{d(T_1 x_{n-1}, T_2 x_n)}$$
$$\le a_1 e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_{n-1}, T_1 x_{n-1})} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_n, T_2 x_n)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_{n-1}, T_2 x_n)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_n, T_1 x_{n-1})}$$
$$= a_1 e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_n, x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_{n-1}, x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_n, x_n)}$$

Again using (2.4) for even n, we obtain

$$e^{d(x_n, x_{n+1})} = e^{d(T_2 x_{n-1}, T_1 x_n)}$$
$$\le a_1 e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_{n-1}, T_2 x_{n-1})} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_n, T_1 x_n)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_{n-1}, T_1 x_n)} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_n, T_2 x_{n-1})}$$
$$= a_1 e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_{n-1}, x_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(x_n, x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_{n-1}, x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(x_n, x_n)}$$

From the above two cases, one can see that

$$e^{d(x_n,x_{n+1})} \le a_1 e^{d(x_{n-1},x_n)} + \left(\frac{a_2+a_3}{2}\right)e^{d(x_{n-1},x_n)} + \left(\frac{a_2+a_3}{2}\right)e^{d(x_n,x_{n+1})} + \left(\frac{a_4+a_5}{2}\right)e^{d(x_{n-1},x_{n+1})} + \left(\frac{a_4+a_5}{2}\right)e^{d(x_n,x_n)}$$

$$\le a_1 e^{d(x_{n-1},x_n)} + \left(\frac{a_2+a_3}{2}\right)e^{d(x_{n-1},x_n)} + \left(\frac{a_2+a_3}{2}\right)e^{d(x_n,x_{n+1})} + \left(\frac{a_4+a_5}{2}\right)e^{d(x_{n-1},x_n)} + \left(\frac{a_4+a_5}{2}\right)e^{d(x_n,x_{n+1})}$$

It follows that $e^{d(x_n,x_{n+1})} \le \dfrac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5} e^{d(x_{n-1},x_n)}$

$$= r e^{d(x_{n-1},x_n)}$$

$\le r^n e^{d(x_0,x_1)} \to 0$ As $n \to \infty$ since r < 1, owing to the assumption $\sum_{i=1}^{5} a_i < 1$

Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let m > n where $m, n \in N$ without any loss of concepts, here two cases arises-

(i) m is even when n is odd.

(ii) m is odd when n is even.

Case I: We choose n and m to be odd & even respectively

Then we have

$$e^{d(x_n,x_m)} = e^{d(T_1x_{n-1},T_2x_{m-1})}$$

$$\le a_1 e^{d(x_{n-1},x_{m-1})} + a_2 e^{d(x_{n-1},T_1x_{n-1})} + a_3 e^{d(x_{m-1},T_2x_{m-1})} + a_4 e^{d(x_{n-1},T_2x_{m-1})} + a_5 e^{d(x_{m-1},T_1x_{n-1})}$$

$$= a_1 e^{d(x_{n-1},x_{m-1})} + a_2 e^{d(x_{n-1},x_n)} + a_3 e^{d(x_{m-1},x_m)} + a_4 e^{d(x_{n-1},x_m)} + a_5 e^{d(x_{m-1},x_n)}$$

Case II: We choose n and m to be even & odd respectively

Then we have

$$e^{d(x_n,x_m)} = e^{d(T_2x_{n-1},T_1x_{m-1})}$$

$$\le a_1 e^{d(x_{n-1},x_{m-1})} + a_2 e^{d(x_{n-1},T_2x_{n-1})} + a_3 e^{d(x_{m-1},T_1x_{m-1})} + a_4 e^{d(x_{n-1},T_1x_{m-1})} + a_5 e^{d(x_{m-1},T_2x_{n-1})}$$

$$= a_1 e^{d(x_{n-1},x_{m-1})} + a_2 e^{d(x_{n-1},x_n)} + a_3 e^{d(x_{m-1},x_m)} + a_4 e^{d(x_{n-1},x_m)} + a_5 e^{d(x_{m-1},x_n)}$$

From above two cases, we get-

$$e^{d(x_n,x_m)} \le a_1 e^{d(x_{n-1},x_{m-1})} + a_2 e^{d(x_{n-1},x_n)} + a_3 e^{d(x_{m-1},x_m)} + a_4 e^{d(x_{n-1},x_m)} + a_5 e^{d(x_{m-1},x_n)}$$

$$\le a_1 e^{d(x_{n-1},x_n)} + a_1 e^{d(x_n,x_m)} + a_1 e^{d(x_m,x_{m-1})} + a_2 e^{d(x_{n-1},x_n)} + a_3 e^{d(x_{m-1},x_m)} + a_4 e^{d(x_n,x_m)} + a_4 e^{d(x_{n-1},x_n)} + a_5 e^{d(x_{m-1},x_m)} + a_5 e^{d(x_m,x_n)}$$

Therefore

$$e^{d(x_n,x_m)} \le \frac{a_1+a_2+a_4}{1-a_1-a_4-a_5} e^{d(x_{n-1},x_n)} + \frac{a_1+a_3+a_5}{1-a_1-a_4-a_5} e^{d(x_{m-1},x_m)}$$

$$\leq \frac{a_1+a_2+a_4}{1-a_1-a_4-a_5} r^{n-1} e^{d(x_0,x_1)} + \frac{a_1+a_3+a_5}{1-a_1-a_4-a_5} r^{m-1} e^{d(x_0,x_1)}$$

$\to 0$ As $n, m \to \infty$, since r < 1.

Hence $\{x_n\}$ is a Cauchy sequence in the complete metric space X, so it is convergent in X.
Let its limit be z, i.e. $\lim_{n\to\infty} x_n = z$. We show that $T_1 z = T_2 z = z$.

Now we have

$$\begin{aligned} e^{d(x_{2n},T_1 z)} &= e^{d(T_2 x_{2n-1},T_1 z)} \\ &\leq a_1 e^{d(x_{2n-1},z)} + a_2 e^{d(x_{2n-1},T_2 x_{2n-1})} + a_3 e^{d(z,T_1 z)} + a_4 e^{d(x_{2n-1},T_1 z)} + a_5 e^{d(z,T_2 x_{2n-1})} \\ &= a_1 e^{d(x_{2n-1},z)} + a_2 e^{d(x_{2n-1},x_{2n})} + a_3 e^{d(z,T_1,z)} + a_4 e^{d(x_{2n-1},T_1 z)} + a_5 e^{d(z,x_{2n})} \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$e^{d(z,T_1 z)} \leq a_3 e^{d(z,T_1 z)} + a_4 e^{d(z,T_1 z)}$$
$$\Rightarrow e^{d(z,T_1 z)} = 0 .$$
$$\Rightarrow z = T_1 z$$

Similarly, it may be shown that $T_2 z = z$ Thus T_1 and T_2 have a common fixed point. For uniqueness, if possible, let w be another common fixed point of T_1 and T_2 such that $w \neq z$.

Now we have

$$\begin{aligned} e^{d(z,w)} &= e^{d(T_1 z,T_2 w)} \\ &\leq a_1 e^{d(z,w)} + a_2 e^{d(z,T_1 z)} + a_3 e^{d(w,T_2 w)} + a_4 e^{d(z,T_2 w)} + a_5 e^{d(w,T_1 z)} \\ &= a_1 e^{d(z,w)} + a_4 e^{d(z,w)} + a_5 e^{d(w,z)} \end{aligned}$$

$\Rightarrow e^{d(z,w)} = 0$, A contradiction. Hence, z = w.

Thus T_1 and T_2 have a unique common fixed point. This completes the proof.

Corollary 2.2: Let (X, d) be a complete metric space. Let a, b, c be positive real numbers satisfying a + b + c < 1, T_1 and T_2 be a pair of self maps of the metric space X into itself such that for each $x, y \in X$,

$$e^{d(T_1 x,T_2 y)} \leq a e^{d(x,T_1 x)} + b e^{d(y,T_2 y)} + c e^{d(x,y)} \qquad \text{.......(2.5)}$$

Where $\phi : R^+ \to R^+$ is a Lebesgue- Integrable map which is summable, positive and such that $e^{\varepsilon} > 0$ for each $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in X$.

Proof: Since the contractive condition (2.5) is obviously a special case of (2.1) by setting $a_1 = c, a_2 = a, a_3 = b$ and $a_4 = a_5 = 0$, the result follows immediately from Theorem (2.1).

Corollary 2.3: Let (X, d) be a complete metric space. Let a, b, c be positive real numbers satisfying a + b + c < 1, T_1 and T_2 be a pair of self maps of the metric space X into itself such that for each $x, y \in X$,

$$e^{d(T_1 x,T_2 y)} \leq a e^{d(x,T_2 y)} + b e^{d(y,T_1 x)} + c e^{d(x.y)} \qquad \text{........ (2.6)}$$

Where $\phi : R^{+} \rightarrow R^{+}$ is a Lebesgue- Integrable map which is summable, positive and such that $e^{\varepsilon} > 0$ for each $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in X$.

Proof: Since the contractive condition (2.6) is obviously a special case of (2.1) by setting $a_1 = c, a_4 = a, a_5 = b$ and $a_2 = a_3 = 0,$ the result follows immediately from Theorem (2.1).

Remark 2.4: We give some remarks which clarify the connection between our results and the results obtained in [2]

(i) Theorem 1 and 2 (cf. [2]) are special cases of corollary 2.2 and 2.3 respectively with $T_1 = T_2$, a = b and c = 0.
(ii) By taking $T_1 = T_2$, Corollary 2.2 and 2.3 reduce Theorem 3 and 4 (cf. [2]) respectively.
(iii) Theorem 5 (cf. [2]) is a consequence of Theorem 2.1 if we take $T_1 = T_2$.

REFERENCES

[1] Vahid Reza Hosseini, "Common fixed point theorems for maps altering distance under a contractive condition of integral type", International Journal Contemporary. Math. Sciences, Vol. 5, No. 33, pp. 1615 – 1620, 2010. (journal style)
[2] Debashis Dey, Anamika Ganguly and Mantu Saha, "Fixed point theorems for mappings under general contractive condition of integral type", Bulletin of Mathematical Analysis and Applications, Vol. 3, Issue No. 1, pp. 27- 34, 2011. (journal style)
[3] Ishak Altun, "Common fixed point theorem for maps satisfying a general contractive condition of integral type", Acta Universitatis Apulensis Vol. No. 22, pp. 195-206, 2010. (journal style)
[4] Massoumeh Hosseinjani Zamenjani, Vahid Reza Hosseini, "Common fixed point theorems for maps altering distance under a contractive condition of integral type for pairs of some compatible maps", International Journal of Math. Analysis, Vol. 6, No. 23, pp. 1123 – 1130, 2012. (journal style)
[5] Sanjay Kumar, Renu Chugh and Ramesh Kumar, "Fixed point theorem for compatible mappings satisfying a contractive condition of integral type", Soochow Journal of Mathematics, Volume 33, No. 2, pp. 181-185, 2007. (journal style)
[6] Manish Kumar Mishra, Priyanka Sharma and D.B. Ojha, "On common fixed point theorems in fuzzy metric spaces satisfying integral type inequality", Research Journal of Applied Sciences, Engineering and Technology, Vol. 2 (8) , pp. 727-733, 2010. (journal style)
[7] M. R. Singh, L. Sharmeswar Singh, "Fixed point theorem for a pair of self maps satisfying a general contractive condition of integral type", Kathmandu University Journal of Science Engineering and Technology, Vol. 6, pp. 20-27, 2010. (journal style)
[8] Dr. Vishal Gupta, Naveen Mani, Naveen Gulati, "A common fixed point theorem satisfying contractive condition of integral type", International Journal of Engineering, Applied Science, Vol. 02, Issue- 2, 2012. (journal style)
[9] Hemant Kumar Pathak, Rakesh Tiwari, Mohammad Saeed Khan, "A common fixed point theorem satisfying integral type implicit relations", Applied Mathematics E-Notes, Vol.7, pp. 222-228, 2007. (journal style)
[10] P. Vijayraju, B. E. Rhoades and R. Mohanraj, "A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type", International Journal of Mathematics and Mathematical Sciences Vol.15, pp. 2359–2364, 2005. (journal style)
[11] B.E. Rhoades, M. Abbas, "Maps satisfying generalised contractive conditions of integrals type", International Journal of Pure and Applied Mathematics Vol. 45 No. 2, pp. 225-23, 2008. (journal style)
[12] Zeqing Liu1, Haijiang Dong, "Fixed and periodic points for mappings satisfying contractive conditions of integral type", International Journal of Pure and Applied Mathematics, Vol. 76 No. 3, pp. 355-369, 2012. (journal style)
[13] Ramakant Bhardwaj, "Fixed point theorems for compatible mapping satisfying a contractive condition of integral type", International Journal of Mathematical Archive Vol. 03 (5), pp. 2157-2162, 2012. (journal style)
[14] M. Abbas and M. Ali Khan, "Common fixed point theorem of two mappings satisfying a generalized weak contractive condition", International Journal of Mathematics and Mathematical Sciences, 2009. (journal style)
[15] O. A. Boikanyo, "Some fixed point theorems for mappings satisfying a general contractive condition of integral types", Far East J. Math. Sci., (FJMS) Vol. 26 (1), pp. 219 – 230, 2007. (journal style)
[16] Branciari, "A fixed point theorem for mappings satisfying a general contractive condition of integral type", Int. J. Math. Science, Vol. 29, pp. 531 – 536, 2002. (journal style)
[17] E. Rhoades, "Two fixed point theorems for mappings satisfying a general contractive condition of integral type", International Journal Math. Science, Vol. 63, pp. 4007 – 4013, 2003. (journal style)

AUTHOR

Priyanka Sharma received the B.Sc. and M.Sc. degrees in Mathematics from CCS University, Meerut (India) in 2007 and 2009 respectively. She is currently working towards the Ph.D. degree at the Department of Mathematics, Mewar University, Chittorgarh, Rajasthan (India). Her current research area is applications of common fixed point theorems.